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Hausdorff Matching and Lipschitz Optimization

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Abstract. In this paper, we prove the Lipschitz continuity with respect to the Hausdorff metric of some parametrized families of sets in R^3. This implies that many Hausdorff approximation (Hausdorff matching) problems can be reduced to searching a global minimum of a real Lipschitz function of real variables. Practical methods are presented for obtaining reduced search spaces for these minimization problems.

Key words: Branch and Bound, Hausdorff approximation, Hausdorff matching, Lipschitz functions

1. Introduction

Shape recognition is a challenging task which has numerous applications. Given a set of model shapes and an input shape, the problem consists of identifying the model shapes which are similar to the input shape. For example, most robotics applications for part inspection and VLSI design involve locating and identifying objects. In general, shape recognition is formulated as the problem of optimizing a cost function. In the case of 2D and 3D problems cost functions frequently used are Hausdorff metric and the volume of the symmetric difference between sets. This paper is about the Hausdorff distance.

Let (X, d) be a complete metric space. We denote by $\mathcal{H}(X)$ the space whose points are the compact subsets of X other than the empty set. If A and B are elements of $\mathcal{H}(X)$ the distance between them is given by

$$d(A, B) \equiv \min\{d(x, y)/x \in A, y \in B\}.$$

Let $x \in X$ and $B \in \mathcal{H}(X)$. Define

$$d(x, B) \equiv \min\{d(x, y)/y \in B\}.$$

Let $A, B \in \mathcal{H}(X)$. Define

$$h(A, B) \equiv \max\{d(x, B)/x \in A\}.$$

$d(x, B)$ is called the distance from the point x to the set B and $h(A, B)$ is called the directed Hausdorff distance from A to B.

The Hausdorff distance between two elements A and B of $\mathcal{H}(X)$ is defined as

$$H(A, B) \equiv \max\{h(A, B), h(B, A)\}.$$

H is a metric on the space $\mathcal{H}(X)$.

From the above definitions, we can give an abstract formulation of the problem of shape recognition based on the Hausdorff distance. Let $\mathcal{F}$ be a family of sets in $\mathcal{H}(X)$ and K^* a fixed element in $\mathcal{H}(X)$. $F \in \mathcal{F}$ is called an element of best approximation of K^* by the elements of $\mathcal{F}$, if

$$H(K^*, F) = \inf_{K \in \mathcal{F}} H(K^*, K). \tag{1}$$

Hausdorff approximation (matching) of a compact set K^* by means of a family $\mathcal{F}$ consists of finding a solution to problem (1).

In R^N, if we denote by $\{M(K_0)\}$ the set of images of K_0 under rigid motions M, the problem of the recognition of the body K^* by means of a set of models $\{K_1, K_2, \dots, K_n\}$, consists of solving problem (1) with $\mathcal{F} = \bigcup_{i=1}^{n} \{M(K_i)\}$. If the solution $F \in \{M(K_j)\}$ we say that K_j is the model corresponding to K^*.

Algorithms for some specific problems have been proposed to achieve exact Hausdorff matching

- In R^2
  - Huttenlocher et al. [17] describe an algorithm that optimally matchs, under translation, a set A of m points onto a set B of n points in time $O(nm(n+m)\log(nm))$.
  - Chew et al. [7] give an algorithm that optimally matchs, under Euclidean motion, a set A of m points onto a set B of n points in time $O((n+m)^5 \log^2(nm))$. They also give an algorithm with expected time $O((n+m)^6 \log^2(nm))$ in the case that A and B are polygons having m and n vertices, respectively. These algorithms are based on the Parametric Search technique.
- In R^3
  - Huttenlocher et al. [17] describe an algorithm that optimally matchs a finite set of m points on a finite set of n points under translation, in time $O(n^2 m^2 (n+m)^{1+\epsilon})$.
  - Zhu [31] gives a method that finds the optimal approximate ball to a convex polyhedron $\mathcal{P}$ with n vertices, in time $O(n^7 \log n)$.

- In R^N
  - Algorithms for set of points and matching under translation are given in [8].
  - Wenk [30] shows that a translation that minimizes the Hausdorff distance between two polyhedral sets of total complexity n in R^{N+1} can be computed in $O(n^{N^2+3N+2}\ \log^2 n)$ time for $N \geqslant 2$.
  - Algorithms for matching two polyhedral terrains in higher dimensions under translation, are given in [18].

Algorithms for finding the optimal solution of the matching problem use quite sophisticated techniques from computational geometry and therefore, are probably too complicated to implement. Also the asymptotic running times are rather high. One approach to overcome these problems are approximation algorithms. These algorithms do not necessarily find the optimal solution ϵ_{opt}, but one which is bounded by ϵ_{opt} multiplied by a constant factor (pseudooptimal matching).

- In R^2
  - Goodrich et al. [11] describe an algorithm that finds a rigid motion M such that $h(M(A), B) \leqslant 4\epsilon_{\mathrm{opt}}$, where A is a set of m points and B a set of n points. The asymptotic running time of this algorithm is $O(n^2 m \log n)$. A more effective method for this problem can be found in [10].
  - Alt et al. [2, 3] describe algorithms for Hausdorff matching of polygons based on the technique of reference points.
- In R^3
  - Goodrich et al. [11] describe an algorithm that finds a rigid motion M such that $h(M(A), B) \leqslant (8+\epsilon)\epsilon_{\mathrm{opt}}$ $(0<\epsilon<1)$, where A is a set of m points and B a set of n points. The asymptotic running time of this algorithm is $O(n^3 m \log n)$. A slightly better algorithm for this problem has been reported in [10].
  - Zhu [31] describes a linear time algorithm that finds an axis-parallel box C such that $H(\mathcal{P}, C) \leqslant \sqrt{3}\epsilon_{\mathrm{opt}}$, where $\mathcal{P}$ is a fixed convex polyhedron with n vertices.
- In R^N
  - Pseudooptimal matching with reference points is studied in [1].
  - Hagedoorn [13] describes a technique called Geometric Branch and Bound (GBB), that recursively subdivides the transformation space into cells. For each cell a lower and upper bounds of the infimum value of the similarity measure are obtained and the convergence of the procedure is proved. The main drawback of this algorithm

is the way of obtaining the cited bounds based on "the traces approach". This method gives an expression of the lower bound for a cell that seems difficult to compute.

Consequently, we can conclude that do not exist efficient algorithms for matching 3D figures. Furthermore, some problems of this kind are NP-hard [4].

With these results in sight, it would be interesting to study the application of standard global optimization techniques to the Hausdorff matching problem.

Here, we consider the following families of sets $\mathcal{F}$, whose elements are subsets of R^3

- ($\mathcal{A}$) The set of axis-parallel boxes of variable center and dimensions.
- ($\mathcal{B}$) The set of spheres of variable center and radius.
- ($\mathcal{C}$) $\{M(K_0)\}$, images of a fixed compact set K_0 under rigid motions M.

We prove that matching problems involving the above families of sets can be solved with an arbitrary precision using standard techniques of global Lipschitz optimization [14]. An important example of them is the Lipschitz branch and bound [16].

We provide explicit upper bounds of the Lipschitz constant for the set families above cited.

This paper is organized as follows:

In Section 2, we define $\mathcal{F}$ as a parametrized family of sets, and show that if this family is Lipschitz continuous with respect to the Hausdorff metric, then the Hausdorff distance from the elements of $\mathcal{F}$ to a fixed $K^* \in \mathcal{H}(X)$ is a Lipschitz function with respect to the parameters of $\mathcal{F}$.

In Section 3, we give auxiliary results about the computation of the Hausdorff distance between different geometric bodies.

In Section 4, we prove that the set families $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are Lipschitz continuous with respect to the Hausdorff metric.

In Section 5, we analyze the problem of determining a reduced search space for the above set families when K^* and the elements of $\mathcal{F}$ are convex. The aim is to reduce this set as much as possible for decreasing the computation time of minimization algorithms based on partition strategies.

From now on, we denote by R^N the real affine euclidean space and also the N-dimensional vectorial euclidean space. We call $\|\cdot\|$ to the euclidean norm in R^N, and d to the distance induced by this norm (sometimes, d will denote an abstract metric). We call $\mathbf{x}.\mathbf{y}$ to the inner product of the vectors $\mathbf{x}$ and $\mathbf{y}$. The translation operator along a vector $\mathbf{u}$ will be denoted by $T_{\mathbf{u}}$.

If S is a bounded subset of R^N we call ∂S to its boundary. The diameter of S is defined as

$$\delta(S) \equiv \sup_{\mathbf{x}_1, \mathbf{x}_2 \in S} d(\mathbf{x}_1, \mathbf{x}_2).$$

If $A \in \mathcal{H}(R^N)$ is convex, we define the circumsphere of A as the sphere of minimum diameter containing A [20, pp. 103]. We denote by $\pi_A(\mathbf{x})$ the Euclidean projection of a point $\mathbf{x}$ onto A.

If $K \in \mathcal{H}(R^N)$ is not convex, we define

$$\pi_K(\mathbf{x}) \equiv \{\mathbf{y} \in K / d(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{z} \in K} d(\mathbf{x}, \mathbf{z})\}.$$

2. Hausdorff Matching and Parametrized Set Families

In this section, we assume that (X, d) is a complete metric space. If $\mathcal{F}$ is a parametrized set family (set valued function), that is,

$$\mathcal{F} \equiv \{K(\mathbf{s}) \in \mathcal{H}(X),\ \mathbf{s} \in D \in \mathcal{H}(R^m)\},$$

problem (1) can be stated as

Find $\mathbf{t} \in D$ such that

$$H(K^*, K(\mathbf{t})) = \inf_{\mathbf{s} \in D} H(K^*, K(\mathbf{s})).$$

$H(\mathbf{s}) \equiv H(K, K(\mathbf{s}))$ can be considered as a function from $D \subset R^m$ into R.

We say that the parametrized family $\mathcal{F}$ is Lipschitz continuous on D with respect to the Hausdorff metric, if there exists a constant $L > 0$ such that

$$H(K(\mathbf{s}), K(\mathbf{s}')) \leqslant L\|\mathbf{s} - \mathbf{s}'\| \quad \text{for all } \mathbf{s}, \mathbf{s}' \in D.$$

LEMMA 1. *Let* $A, B, C \in \mathcal{H}(X)$, *then*

$$|H(A, B) - H(A, C)| \leqslant H(B, C).$$

Proof. Since the Hausdorff distance is a metric on $\mathcal{H}(X)$ [6, pp. 32], we have

$$H(A, B) \leqslant H(A, C) + H(C, B),$$
$$H(A, C) \leqslant H(A, B) + H(B, C).$$

□

Hence

PROPOSITION 1. *If $\mathcal{F}$ is Lipschitz continuous on D, then $H(\mathbf{s})$ is a Lipschitz function on D.*

Proof. By Lemma 1, if $\mathbf{s}, \mathbf{s}' \in D$

$$\begin{aligned}|H(\mathbf{s}) - H(\mathbf{s}')| &= |H(K, K(\mathbf{s})) - H(K, K(\mathbf{s}'))| \\ &\leqslant H(K(\mathbf{s}), K(\mathbf{s}')) \leqslant L\|\mathbf{s} - \mathbf{s}'\|.\end{aligned}$$

□

Therefore, we can solve Hausdorff matching problems of the above type by means of an algorithm based on the Lipschitz property [14], if the following two conditions are verified:

- (L1) $H(K^*, K(\mathbf{s}))$ can be computed for all $\mathbf{s} \in D$.
- (L2) The family $\mathcal{F} = \{K(\mathbf{s}) \in \mathcal{H}(X)/\mathbf{s} \in D\}$ is Lipschitz continuous on D.

The parametrized set families $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ do not have a compact domain for their parameters. In Section 5, we show how to obtain a suitable compact subdomain depending on the set K^* to be approximated.

3. Computation of the Hausdorff Distance in Some Particular Cases

In this section, we give expressions based on projections for the calculus of the Hausdorff distance in some particular cases. They will be useful for

- Computing the succesive Hausdorff distances in a Lipschitz optimization procedure (L1).
- Proving the Lipschitz continuity of the parametrized set families $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ (L2), and finding concrete upper bounds of the Lipschitz constant (Section 4).
- Reducing the search space of the minimization problems (Section 5).

We need some previous definitions and results.

Let S be a set contained in R^N, we call convex hull of S to

$$\mathrm{conv}(S) = \bigcap \{C/C \text{ is convex and } C \supset S\}.$$

We call polytope to the convex hull of a finite set of points in R^N.

The set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ is a minimal representation of the polytope $\mathcal{P}$, if $\mathcal{P} = \mathrm{conv}(\{\mathbf{x}_1, \ldots, \mathbf{x}_p\})$ and for each $i = 1, \ldots, p$

$$\mathbf{x}_i \notin \mathrm{conv}\left(\bigcup_{j \neq i} \mathbf{x}_j\right).$$

Every polytope has a minimal representation and this representation is unique [20]. We call vertices of $\mathcal{P}$ to the points $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ of a minimal representation of $\mathcal{P}$.

LEMMA 2. *Let $\mathcal{P}$ be a polytope defined as the convex hull of their vertices, that is, $\mathcal{P} = \mathrm{conv}(\{\mathbf{x}_1, \dots, \mathbf{x}_p\})$. Let K be a compact and convex set in R^N, then*

$$h(\mathcal{P}, K) = \max_{i=1\dots p} d(\mathbf{x_i}, K).$$

From this result, we have

LEMMA 3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes defined as the convex hull of their vertices, that is,*

$$\mathcal{P} = \mathrm{conv}(\{\mathbf{x}_1, \dots, \mathbf{x}_p\}), \qquad \mathcal{Q} = \mathrm{conv}(\{\mathbf{y}_1, \dots, \mathbf{y}_q\})$$

then, we have

$$H(\mathcal{P}, \mathcal{Q}) = \max\left(\max_{i=1,\dots,p} d(\mathbf{x}_i, \mathcal{Q}), \max_{j=1,\dots,q} d(\mathbf{y}_j, \mathcal{P}) \right).$$

The above formula can be written in the operative form

$$H(\mathcal{P}, \mathcal{Q}) = \max\left(\max_{i=1,\dots,p} d(\mathbf{x}_i, \pi_{\mathcal{Q}}(\mathbf{x}_i)), \max_{j=1,\dots,q} d(\mathbf{y}_j, \pi_{\mathcal{P}}(\mathbf{y}_j)) \right). \tag{2}$$

A proof of Lemma 2 can be found in [25] (in that paper is given an efficient algorithm for finding the Hausdorff distance between polytopes which have many vertices).

The projections in (2) can be calculated by means of projectors on polytopes [23, 29], or fast projectors on polyhedra based on local properties [22, 24].

To obtain the Hausdorff distance between two spheres we use

LEMMA 4. *Let (X, d) be a complete metric space and let $A, B, C \in \mathcal{H}(X)$. If $B \subset C$, then $h(A, C) \leqslant h(A, B)$.*

For a proof, see [6, pp. 29].

LEMMA 5. *Let* $S_1 = \bar{B}_{r_1}(\mathbf{c}_1)$ *and* $S_2 = \bar{B}_{r_2}(\mathbf{c}_2)$ *be two closed spheres in* R^N *and the sets*

$$A_i = \{\mathbf{c}_2 + \lambda(\mathbf{c}_1 - \mathbf{c}_2), \lambda \geqslant 0\} \cap \partial S_i, \quad i = 1, 2.$$
$$B_1 = \{\mathbf{c}_1 + \lambda(\mathbf{c}_2 - \mathbf{c}_1), \lambda \geqslant 0\} \cap \partial S_2,$$
$$B_2 = \{\mathbf{c}_1 + \lambda(\mathbf{c}_2 - \mathbf{c}_1), \lambda \geqslant 0\} \cap \partial S_1.$$

Consider the points

- $\mathbf{a}_1, \mathbf{b}_1$, *elements of* A_1 *and* B_1, *respectively, with maximum value of* λ.
- $\mathbf{a}_2 = A_2$ *and* $\mathbf{b}_2 = B_2$.

Then

$$h(S_1, S_2) = \|\mathbf{a}_1 - \mathbf{a}_2\| \ \ (0, \ if \ S_1 \subset S_2),$$
$$h(S_2, S_1) = \|\mathbf{b}_1 - \mathbf{b}_2\| \ \ (0, \ if \ S_2 \subset S_1).$$

Proof. To establish the identity for $h(S_1, S_2)$ we suppose that S_1 is not contained in S_2. Consider the following two cases:

(1) $\mathbf{c}_1 \notin S_2$. We have that $\mathbf{a}_2 = \pi_{S_2}(\mathbf{c}_1)$. Let $\mathbf{a}'$ be any other point in $S_1 - S_2$. We have to prove that

$$\|\mathbf{a}' - \pi_{S_2}(\mathbf{a}')\| \leqslant \|\mathbf{a}_1 - \pi_{S_2}(\mathbf{c}_1)\| = \|\mathbf{a}_1 - \mathbf{a}_2\|.$$

By the properties of the projection on a closed, convex set [5, pp. 11]

$$(\pi_{S_2}(\mathbf{a}') - \pi_{S_2}(\mathbf{c}_1)).(\mathbf{a}' - \pi_{S_2}(\mathbf{a}')) \geqslant 0. \tag{3}$$

On the other hand

$$\|\mathbf{a}' - \mathbf{c_1}\| \leqslant \|\mathbf{a_1} - \mathbf{c_1}\| = r_1. \tag{4}$$

Since

$$(\pi_{S_2}(\mathbf{a}') - \pi_{S_2}(\mathbf{c}_1)) + (\mathbf{a}' - \pi_{S_2}(\mathbf{a}')) = (\mathbf{c_1} - \pi_{S_2}(\mathbf{c_1})) + (\mathbf{a}' - \mathbf{c}_1)$$

it follows that

$$\|\pi_{S_2}(\mathbf{a}') - \pi_{S_2}(\mathbf{c}_1)\|^2 + \|\mathbf{a}' - \pi_{S_2}(\mathbf{a}')\|^2 + 2(\pi_{S_2}(\mathbf{a}') - \pi_{S_2}(\mathbf{c}_1)).(\mathbf{a}' - \pi_{S_2}(\mathbf{a}'))$$
$$= \|\mathbf{c}_1 - \pi_{S_2}(\mathbf{c_1})\|^2 + \|\mathbf{a}' - \mathbf{c}_1\|^2 + 2(\mathbf{c}_1 - \pi_{S_2}(\mathbf{c}_1)).(\mathbf{a}' - \mathbf{c}_1).$$

Applying the Schwarz inequality and using (3) and (4), we have

$$\begin{aligned}\|\mathbf{a}' - \pi_{S_2}(\mathbf{a}')\|^2 &\leqslant (\|\mathbf{c}_1 - \pi_{S_2}(\mathbf{c}_1)\| + \|\mathbf{a}' - \mathbf{c}_1\|)^2 \\ &\leqslant (\|\mathbf{c}_1 - \pi_{S_2}(\mathbf{c}_1)\| + \|\mathbf{a}_1 - \mathbf{c}_1\|)^2 = \|\mathbf{a}_1 - \pi_{S_2}(\mathbf{c}_1)\|^2.\end{aligned}$$

(2) $\mathbf{c}_1 \in S_2$. In this case, $\mathbf{a}_2$ is the nearest point to $\mathbf{c}_1$ in ∂S_2. If we call d_m to the distance $d(\mathbf{a}_2, \mathbf{c}_1)$ we have $h(S_1, S_2) \geqslant r_1 - d_m$, $\bar{B}_{d_m}(\mathbf{c}_1) \subset S_2$ and by Lemma 4, we have

$$r_1 - d_m \leqslant h(S_1, S_2) \leqslant h(S_1, \bar{B}_{d_m}(\mathbf{c}_1)) = r_1 - d_m.$$

Therefore,

$$h(S_1, S_2) = r_1 - d_m = \|\mathbf{a}_1 - \mathbf{a}_2\|.$$

$h(S_2, S_1)$ is obtained by analogous reasoning. □

The Hausdorff distance between a convex polytope and a sphere is given by

PROPOSITION 2. *Let S be a sphere in R^N of radius r and center $\mathbf{c}$ ($S = \bar{B}_r(\mathbf{c})$), and let $\mathcal{P}$ be a polytope defined as the convex hull of their vertices, that is, $\mathcal{P} = \mathrm{conv}(\{\mathbf{x}_1, \ldots, \mathbf{x}_p\})$, then*

1. $h(\mathcal{P}, S) = \max_{i=1,\ldots,p} d(\mathbf{x}_i, S)$.
2. *If $\mathbf{c} \notin \mathcal{P}$. Consider the set*

$$A = \{\pi_{\mathcal{P}}(\mathbf{c}) + \lambda(\mathbf{c} - \pi_{\mathcal{P}}(\mathbf{c})), \lambda \geqslant 0\} \cap \partial S.$$

Let $\mathbf{a}$ be the element of A with maximum λ, then

$$h(S, \mathcal{P}) = \|\mathbf{a} - \pi_{\mathcal{P}}(\mathbf{c})\|.$$

3. *If $\mathbf{c} \in \mathcal{P}$. Let d_m be the shortest distance from $\mathbf{c}$ to the faces of $\mathcal{P}$, then*
   - *If $R > d_m$, then $h(S, \mathcal{P}) = r - d_m$.*
   - *If $R \leqslant d_m$, then $h(S, \mathcal{P}) = 0$.*

Proof. Part 1 is a consequence of Lemma 2. Parts 2 and 3 can be proved in a similar way than Lemma 5. □

We remark that the above proposition can be stated if the set $\mathcal{P}$ is any convex and compact set in R^N. The difference is

$$h(\mathcal{P}, S) = \max_{\mathbf{x} \in \mathcal{P}} d(\mathbf{x}, S)$$

and

$$d_m = d(\mathbf{c}, \partial\mathcal{P}).$$

4. Lipschitz Continuity of Some Parametrized Set Families in R^3

4.1. AXIS-PARALLEL BOXES OF VARIABLE CENTER AND DIMENSIONS

Let $\mathbf{a}, \mathbf{a}' \in R^3$ and let $\mathbf{d}, \mathbf{d}' \in R^3_+ \equiv \{\mathbf{x} \in R^3 / x_i \geqslant 0, i = 1, 2, 3\}$. Two axis-parallel boxes C_1 and C_2 are defined by

$$C_1 = \operatorname{conv}(\{\mathbf{a} + \mathbf{b}.\mathbf{d} / \mathbf{b} \in \{0, 1\}^3\}),$$
$$C_2 = \operatorname{conv}(\{\mathbf{a}' + \mathbf{b}'.\mathbf{d}' / \mathbf{b}' \in \{0, 1\}^3\}).$$

We say that the vertices $\mathbf{v}_k$ of C_1 and $\mathbf{w}_k$ of C_2 are homologous if $N_{C_1}(\mathbf{v}_k) = N_{C_2}(\mathbf{w}_k)$ (where $N_C(\mathbf{x})$ is the normal cone of the convex C at $\mathbf{x}$).

LEMMA 6. *Let $\mathbf{v}_k$ and $\mathbf{w}_k$ be two homologous vertices in C_1 and C_2, respectively, then*

$$d(\mathbf{v}_k, \pi_{C_2}(\mathbf{v}_k)) \leqslant d(\mathbf{v}_k, \mathbf{w}_k).$$

Proof. By definition

$$d(\mathbf{v}_k, \pi_{C_2}(\mathbf{v}_k)) = \min_{\mathbf{w} \in C_2} d(\mathbf{v}_k, \mathbf{w}).$$

□

THEOREM 1. *The family of axis-parallel boxes in R^3 defined by*

$$\mathcal{A} \equiv \{\operatorname{conv}(\{\mathbf{a} + \mathbf{b}.\mathbf{d} / \mathbf{b} \in \{0, 1\}^3\}), \mathbf{a} \in R^3, \mathbf{d} \in R^3_+\}$$

is a Lipschitz continuous function of the six real variables $(\mathbf{a}, \mathbf{d})$.

Proof. Consider two axis-parallel boxes C_1 and C_2. By Lemmas 3 and 6, we have that there exist two homologous vertices $\mathbf{v}_k, \mathbf{w}_k$ in C_1 and C_2, respectively, such that

$$H(C_1, C_2) \leqslant d(\mathbf{v}_k, \mathbf{w}_k).$$

Since $\mathbf{v}_k$ and $\mathbf{w}_k$ are homologous, we have

$$\mathbf{v}_{k_i} = a_i + b_i d_i, \qquad \mathbf{w}_{k_i} = a'_i + b_i d'_i, \quad i = 1, 2, 3.$$

Hence

$$\begin{aligned} d(\mathbf{v}_k, \mathbf{w}_k) &= \sqrt{\sum_{i=1}^{3} (a_i - a'_i + b_i(d_i - d'_i))^2} \\ &\leqslant \sqrt{\sum_{i=1}^{3} 2((a_i - a'_i)^2 + (d_i - d'_i)^2)} \\ &\leqslant \sqrt{2} \sqrt{\sum_{i=1}^{3} ((a_i - a'_i)^2 + (d_i - d'_i)^2)}. \end{aligned}$$

□

4.2. SPHERES OF VARIABLE CENTER AND RADIUS

THEOREM 2. *The family of closed spheres in* R^3

$$\mathcal{B} \equiv \{\bar{B}_r(\mathbf{c}),\ \mathbf{c} \in R^3, r \in R_+\}$$

is a Lipschitz continuous function of the four real variables $(\mathbf{c}, r)$.

Proof. Let $S_1 = \bar{B}_{r_1}(\mathbf{c}_1)$ and $S_2 = \bar{B}_{r_2}(\mathbf{c}_2)$ be two spheres.
If we call $\mathbf{d}$ to $\mathbf{c}_1 - \mathbf{c}_2$, we have

$$\mathbf{a}_1 = \mathbf{c}_1 + r_1 \frac{\mathbf{d}}{\|\mathbf{d}\|}, \qquad \mathbf{a}_2 = \mathbf{c}_2 + r_2 \frac{\mathbf{d}}{\|\mathbf{d}\|}$$

and, by Lemma 5

$$\begin{aligned} h(S_1, S_2) &= \|\mathbf{a}_1 - \mathbf{a}_2\| = \|\mathbf{d} + (r_1 - r_2)\frac{\mathbf{d}}{\|\mathbf{d}\|}\| \\ &\leqslant \|\mathbf{d}\| + |r_1 - r_2| \\ &\leqslant \sqrt{2} \sqrt{\sum_{i=1}^{3} (\mathbf{c}_{1i} - \mathbf{c}_{2i})^2 + (r_1 - r_2)^2}. \end{aligned}$$

(if $S_1 \subset S_2$, then $h(S_1, S_2) = 0$ and the bound remains valid).

We have the same result for $h(S_2, S_1)$, therefore,

$$H(S_1, S_2) \leqslant \sqrt{2}\sqrt{\sum_{i=1}^{3}(\mathbf{c}_{1i} - \mathbf{c}_{2i})^2 + (r_1 - r_2)^2}.$$

□

4.3. SETS GENERATED BY RIGID MOTIONS OF A COMPACT SET K_0 IN R^3

A rigid motion in R^N is defined by the application $\mathbf{y} = R\mathbf{x} + \mathbf{t}$.

- R is an element of the group $SO(N)$ [real orthogonal matrices ($R^t R = I$) such that $det\, R = 1$].
- $\mathbf{t}$ is an element of R^N.

The set of rigid motions in R^N will be denoted by $RM(N)$.
Let K_0 be a fixed compact set in R^3, define

$$\mathcal{C} \equiv \{K \in \mathcal{H}(R^3)/\exists M \in RM(3) \text{ such that } K = M(K_0)\}.$$

We need the following results and definitions.

LEMMA 7. *Let K_0 be a compact set in R^3, let M and M' be rigid motions in R^3, $K = M(K_0)$ and $K' = M'(K_0)$, then*

$$H(K, K') \leqslant \max_{\mathbf{v}\in K_0} \|M(\mathbf{v}) - M'(\mathbf{v})\|.$$

Proof. By definition

$$H(K, K') = \max(\max_{\mathbf{y}\in K} \|\mathbf{y} - \pi_{K'}(\mathbf{y})\|, \max_{\mathbf{z}\in K'} \|\mathbf{z} - \pi_K(\mathbf{z})\|).$$

(if $\pi_{K'}$ and π_K are multivalued, $\pi_{K'}(\mathbf{y})$ and $\pi_K(\mathbf{z})$ denote any fixed representant of the respective set).
Let M_d be the rigid motion $M' \circ M^{-1}$, we have

$$\begin{aligned} H(K, K') &\leqslant \max(\max_{\mathbf{y}\in K} \|\mathbf{y} - M_d(\mathbf{y})\|, \max_{\mathbf{z}\in K'} \|\mathbf{z} - M_d^{-1}(\mathbf{z})\|) \\ &= \max(\max_{\mathbf{v}\in K_0} \|M(\mathbf{v}) - M_d(M(\mathbf{v}))\|, \max_{\mathbf{w}\in K_0} \|M'(\mathbf{w}) - M_d^{-1}(M'(\mathbf{w}))\|) \\ &= \max(\max_{\mathbf{v}\in K_0} \|M(\mathbf{v}) - M'(\mathbf{v})\|, \max_{\mathbf{w}\in K_0} \|M'(\mathbf{w}) - M(\mathbf{w})\|). \end{aligned}$$

□

THEOREM 3. *For every rotation* $R \in SO(3)$ *there are three angles* θ_1, θ_2, *and* θ_3 (*Euler angles*) *such that* R *is a matrix of the form*

$$\begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -s_1c_3 - c_1c_2s_3 & s_2s_3 \\ c_1s_3 + s_1c_2c_3 & -s_1s_3 + c_1c_2c_3 & -s_2c_3 \\ s_1s_2 & c_1s_2 & c_2 \end{bmatrix}$$

(*We denote by* s_i *and* c_i, $\sin\theta_i$ *and* $\cos\theta_i$, *respectively*). *Furthermore*

$$0 \leqslant \theta_1, \theta_3 \leqslant 2\pi, \tag{5}$$
$$0 \leqslant \theta_2 < \pi. \tag{6}$$

For a proof, see [21, pp. 32].

In addition to the euclidean norm we use the following norms in R^N

$$\|\mathbf{x}\|_\infty = \max_{i=1,\dots N} |x_i|.$$
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{N} |x_i|.$$

If A is a matrix associated to a linear transformation from R^N to R^N, the matrix norm subordinated to the vector norm $\|.\|_v$ is defined as

$$\|A\|_v \equiv \max_{\|\mathbf{x}\|_v=1} \|A\mathbf{x}\|_v.$$

A consequence of this definition is

$$\|A\mathbf{x}\|_v \leqslant \|A\|_v \, \|\mathbf{x}\|_v.$$

Some properties of norms are given by

LEMMA 8. *If* $\mathbf{x} \in R^N$ *and* A *is a real,* $N \times N$ *matrix, then*

(*a*) $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\| \leqslant \sqrt{N}\|\mathbf{x}\|_\infty$.

(*b*) $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_1 \leqslant N\|\mathbf{x}\|_\infty$.

(*c*) $\|A\|_\infty = \max_{i=1,\dots N} \sum_{j=1}^{N} |a_{ij}|$.

We use the notation

$$\begin{aligned} \theta &\equiv (\theta_1, \theta_2, \theta_3), \\ \mathbf{t} &\equiv (t_1, t_2, t_3), \\ (\theta, \mathbf{t}) &\equiv (\theta_1, \theta_2, \theta_3, t_1, t_2, t_3). \end{aligned}$$

LEMMA 9. If M and M' are rigid motions in R^3, and $\mathbf{v} \in R^3$, then

$$\|M(\mathbf{v}) - M'(\mathbf{v})\| \leqslant (1 + (6\sqrt{3} + \sqrt{6})\,\|\mathbf{v}\|_\infty)\|(\theta, \mathbf{t}) - (\theta', \mathbf{t}')\|.$$

Proof. By the definition of rigid motion, we have

$$\|M(\mathbf{v}) - M'(\mathbf{v})\| \leqslant \|(R - R')\mathbf{v}\| + \|\mathbf{t} - \mathbf{t}'\|. \tag{7}$$

We use the notation

$$\Delta \equiv R - R'.$$

If we apply the mean value theorem, the Schwarz inequality and the bounds

$$|s_i| \leqslant 1 \quad |c_i| \leqslant 1,$$

to the entries of R, we have

$$\begin{aligned} |\Delta_{11}|, |\Delta_{12}|, |\Delta_{21}|, |\Delta_{22}| &\leqslant 3\|\theta - \theta'\|, \\ |\Delta_{13}|, |\Delta_{23}|, |\Delta_{31}|, |\Delta_{32}| &\leqslant \sqrt{2}\|\theta - \theta'\|, \\ |\Delta_{33}| &\leqslant \|\theta - \theta'\|. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{j=1}^{3} |\Delta_{1j}|, \sum_{j=1}^{3} |\Delta_{2j}| &\leqslant (6 + \sqrt{2})\|\theta - \theta'\|, \\ \sum_{j=1}^{3} |\Delta_{3j}| &\leqslant (1 + 2\sqrt{2})\|\theta - \theta'\|, \end{aligned}$$

therefore, by Lemma 8(c)

$$\|\Delta\|_\infty = \max_{i=1,3}\Big(\sum_{j=1}^{3} |\Delta_{ij}|\Big) \leqslant (6 + \sqrt{2})\|\theta - \theta'\|.$$

By Lemma 8(a)

$$\begin{aligned}\|\Delta\mathbf{v}\| &\leqslant \sqrt{3}\|\Delta\mathbf{v}\|_\infty\\ &\leqslant \sqrt{3}\|\Delta\|_\infty\|\mathbf{v}\|_\infty\\ &\leqslant (6\sqrt{3}+\sqrt{6})\|\mathbf{v}\|_\infty\|\theta-\theta'\|.\end{aligned}$$

Hence, by (7),

$$\begin{aligned}\|M(\mathbf{v})-M'(\mathbf{v})\| &\leqslant \|\Delta\mathbf{v}\|+\|\mathbf{t}-\mathbf{t}'\|\\ &\leqslant (6\sqrt{3}+\sqrt{6})\|\mathbf{v}\|_\infty\,\|\theta-\theta'\|+\|\mathbf{t}-\mathbf{t}'\|\\ &\leqslant (1+(6\sqrt{3}+\sqrt{6})\|\mathbf{v}\|_\infty)\|(\theta,\mathbf{t})-(\theta',\mathbf{t}')\|.\end{aligned}$$

□

THEOREM 4. *The family of compact sets generated by rigid motions of a fixed* $K_0\in\mathcal{H}(R^3)$

$$\mathcal{C}\equiv\{K\in\mathcal{H}(R^3)/\,\exists M\in RM(3)\ such\ that\ K=M(K_0)\}$$

is a Lipschitz continuous function of the six real variables $(\theta,\mathbf{t})$.

Proof. By Lemma 7

$$H(M(K_0),M'(K_0))\leqslant \max_{\mathbf{v}\in K_0}\|M(\mathbf{v})-M'(\mathbf{v})\|$$

hence, by Lemma 9

$$H(M(K_0),M'(K_0))\leqslant \max_{\mathbf{v}\in K_0}(1+(6\sqrt{3}+\sqrt{6})\|\mathbf{v}\|_\infty)\,\|(\theta,\mathbf{t})-(\theta',\mathbf{t}')\|.$$

□

We remark that

- If we repeat the steps of the proof of Lemma 9 for the rotation expressed as a Yaw–Pitch–Roll transformation, we obtain the same upper bound of the Lipschitz constant.
- If K_0 is a polyhedron defined as the convex hull of their vertices, that is, $K_0=\mathrm{conv}(\{\mathbf{x}_1,\dots,\mathbf{x}_p\})$, the upper bound of the Lipschitz constant is

$$\max_{\mathbf{v}\in K_0}(1+(6\sqrt{3}+\sqrt{6})\|\mathbf{v}\|_\infty)=1+(6\sqrt{3}+\sqrt{6})\,(\max_{i=1\dots p}\|\mathbf{x}_i\|_\infty).$$

- If K_0 is a finite set of points, the upper bound is

$$1+(6\sqrt{3}+\sqrt{6})\,(\max_{\mathbf{v}_i\in K_0}\|\mathbf{v}_i\|_\infty).$$

5. Determination of a Reduced Search Domain for Parameters in Hausdorff Matching Problems

When we try to solve a Hausdorff matching problem of type $\mathcal{A}$ or $\mathcal{B}$ by means of a Lipschitz optimization procedure, the initial domain of the parameters is infinite.

In the case of rigid motions ($\mathcal{C}$) the search space for the angular variables $(\theta_1, \theta_2, \theta_3)$ is given by (5) and (6) but the search space for the translation variables (t_1, t_2, t_3) is infinite.

This fact makes difficult the application of partition strategies to Hausdorff matching problems. In this section, we give practical procedures for obtaining reduced search spaces in the case of compact, convex sets. The aim is to decrease this domain as much as possible.

First, from the results of Section 4, we note that the functions

$$H(\mathbf{s}) = H(K^*, K(\mathbf{s})),$$

where $K(\mathbf{s}) \in \mathcal{F}$ $(\mathcal{A}, \mathcal{B}$ or $\mathcal{C})$, are continuous. Since they are also coercive, they have at least one global minimizer [16, pp. 13].

We need the following auxiliary results.

PROPOSITION 3. *Let $A, B \in \mathcal{H}(R^3)$ be convex sets, such that*

$$H(A, B) = \min_{K \in \mathcal{F}} H(A, K),$$

where $\mathcal{F} = \mathcal{A}, \mathcal{B}$ or $\mathcal{C}$, then $d(A, B) = 0$.

Proof. It is equivalent to prove that, if $d(A, B) \neq 0$, then

$$H(A, B) \neq \min_{K \in \mathcal{F}} H(A, K).$$

Let $\mathbf{a}_0 \in A$ and $\mathbf{b}_0 \in B$ be a nearest pair of points in A and B, respectively. By the convexity of A and B, we can find two planes P_A and P_B which are orthogonal to the vector $\mathbf{b}_0 - \mathbf{a}_0$ and that define the halfspaces

$$P_A^- = \{\mathbf{x} \in R^3 / (\mathbf{x} - \mathbf{a}_0).(\mathbf{b}_0 - \mathbf{a}_0) \leqslant 0\},$$
$$P_B^+ = \{\mathbf{x} \in R^3 / (\mathbf{x} - \mathbf{b}_0).(\mathbf{b}_0 - \mathbf{a}_0) \geqslant 0\}$$

and such that

- $\mathbf{a}_0 \in P_A$ and $\mathbf{b}_0 \in P_B$.
- $P_A^- \supset A$ and $P_B^+ \supset B$.
- $d(P_A, P_B) = d(A, B)$.

Let $\mathbf{a}$ be any point of A and $\mathbf{b} = \pi_B(\mathbf{a})$. Translate B towards A along

$$\mathbf{u} = \frac{\mathbf{a}_0 - \mathbf{b}_0}{\|\mathbf{a}_0 - \mathbf{b}_0\|}\gamma \quad (0 < \gamma < d(A, B)).$$

Let $B' = T_{\mathbf{u}}(B)$ and $\mathbf{b}' = T_{\mathbf{u}}(\mathbf{b})$. Take the point $\mathbf{a}_0$ as the coordinate origin, and the line that contains $\mathbf{b}_0 - \mathbf{a}_0$ as the x-axis, we have

$$\mathbf{a} = (a_1, a_2, a_3) \qquad \mathbf{b} = (b_1, b_2, b_3) \qquad \mathbf{b}' = (b_1 - \gamma, b_2, b_3), \tag{8}$$

hence

$$d(\mathbf{a}, B') \leqslant d(\mathbf{a}, \mathbf{b}') < d(\mathbf{a}, \mathbf{b}) \leqslant h(A, B)$$

therefore,

$$h(A, B') = \max_{\mathbf{a} \in A} d(\mathbf{a}, B') < h(A, B).$$

Now, let $\mathbf{b}'$ be any point of B' and $\mathbf{b} = T_{-\mathbf{u}}(\mathbf{b}')$. By (8), we have

$$d(\mathbf{b}', A) \leqslant d(\mathbf{b}', \mathbf{a}) < d(\mathbf{b}, \mathbf{a}) \leqslant h(B, A),$$

hence

$$h(B', A) = \max_{\mathbf{b}' \in B'} d(\mathbf{b}', A) < h(B, A).$$

Therefore,

$$H(A, B') < H(A, B).$$

Finally, consider

- If $B = \text{conv}(\{\mathbf{a} + \mathbf{b}.\mathbf{d}/\mathbf{b} \in \{0, 1\}^3\}) \in \mathcal{A}$, then $T_{\mathbf{u}}(B) = \text{conv}(\{\mathbf{a} + \mathbf{u} + \mathbf{b}.\mathbf{d}/\mathbf{b} \in \{0, 1\}^3\}) \in \mathcal{A}$.
- If $B = \bar{B}_r(\mathbf{c}) \in \mathcal{B}$, then $T_{\mathbf{u}}(B) = \bar{B}_r(\mathbf{c} + \mathbf{u}) \in \mathcal{B}$.
- If $B \in \mathcal{C}$, then $T_{\mathbf{u}}(B) \in \mathcal{C}$.

$\square$

LEMMA 10. *If $A \in \mathcal{H}(R^3)$ is convex, then there exists a unique point $\mathbf{p}_0 \in A$ such that*

$$H(\{\mathbf{p}_0\}, A) = \min_{\mathbf{p} \in R^3} H(\{\mathbf{p}\}, A).$$

Proof. By definition

$$H(\{\mathbf{p}\}, A) = \max\{d(\mathbf{p}, A), \max_{\mathbf{x}\in A} d(\mathbf{p}, \mathbf{x})\}.$$

First, we prove that $\mathbf{p}_0 \notin A$ is impossible. It suffices to prove that if $\mathbf{p} \notin A$, then we can find a point $\mathbf{p}' \in A$ such that

$$H(\{\mathbf{p}'\}, A) < H(\{\mathbf{p}\}, A). \tag{9}$$

Let $\mathbf{p}'$ be $\pi_A(\mathbf{p})$, then

$$0 = d(\mathbf{p}', A) < d(\mathbf{p}, A) = \|\mathbf{p} - \mathbf{p}'\|. \tag{10}$$

If $\mathbf{x} \in A$, from the convexity of A, it follows that

$$d^2(\mathbf{p}, \mathbf{p}') + d^2(\mathbf{p}', \mathbf{x}) \leqslant d^2(\mathbf{p}, \mathbf{x}),$$

hence

$$d(\mathbf{p}', \mathbf{x}) < d(\mathbf{p}, \mathbf{x}) \quad \text{for all } \mathbf{x} \in A. \tag{11}$$

Let $\mathbf{a} \in A$ such that

$$d(\mathbf{p}, \mathbf{a}) = \max_{\mathbf{x}\in A} d(\mathbf{p}, \mathbf{x}) \tag{12}$$

and let $\mathbf{a}' \in A$ such that

$$d(\mathbf{p}', \mathbf{a}') = \max_{\mathbf{x}\in A} d(\mathbf{p}', \mathbf{x}). \tag{13}$$

By (11) and (12)

$$d(\mathbf{p}', \mathbf{a}') < d(\mathbf{p}, \mathbf{a}') \leqslant d(\mathbf{p}, \mathbf{a}). \tag{14}$$

(9) follows from (10) and (12)–(14).
By Lemma 1

$$|H(\mathbf{x}, A) - H(\mathbf{y}, A)| \leqslant \|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in R^3.$$

Therefore, if A is fixed, $H(\mathbf{x}, A)$ is a continuous function of $\mathbf{x}$. Since A is compact, we have that $H(\mathbf{x}, A)$ has a global minimum in A.

Finally, we prove that $\mathbf{p}_0$ is unique. Let $\mathbf{p}_0, \mathbf{p}_0' \in A$ such that

$$\mathbf{p}_0 \neq \mathbf{p}_0'$$

and

$$\min_{\mathbf{p} \in A}(\max_{\mathbf{x} \in A} d(\mathbf{p}, \mathbf{x})) = \max_{\mathbf{x} \in A} d(\mathbf{p}_0, \mathbf{x}) = \max_{\mathbf{x} \in A} d(\mathbf{p}_0', \mathbf{x}) = r, \tag{15}$$

where r is the radius of the lowest volume sphere containing A (circumsphere of A).

Since $A \subset \bar{B}_r(\mathbf{p}_0)$ and $A \subset \bar{B}_r(\mathbf{p}_0')$, it follows that

$$A \subset \bar{B}_r(\mathbf{p}_0) \cap \bar{B}_r(\mathbf{p}_0').$$

Since A is convex, $\mathbf{m} = (\mathbf{p}_0 + \mathbf{p}_0')/2 \in A$. We can choose the coordinate system in such a way that $\mathbf{m} = (0, 0, 0)$, $\mathbf{p}_0 = (\gamma, 0, 0)$, and $\mathbf{p}_0' = (-\gamma, 0, 0)$, where $0 < \gamma < r$, then

- If $\gamma < \frac{r}{\sqrt{2}}$ then $A \subset \bar{B}_r(\mathbf{p}_0) \cap \bar{B}_r(\mathbf{p}_0') \subset \bar{B}_{\sqrt{r^2 - \gamma^2}}(\mathbf{m})$.
- If $\gamma \geqslant \frac{r}{\sqrt{2}}$ then $A \subset \bar{B}_r(\mathbf{p}_0) \cap \bar{B}_r(\mathbf{p}_0') \subset \bar{B}_\gamma(\mathbf{m})$.

Then, if $\mathbf{p}_0 \neq \mathbf{p}_0'$ we can find a sphere centered at a point of A with radius strictly lower than r, and containing A. This contradicts (15).

□

In practice, we have to compute the circumcenter of a given convex and compact set A, that is, to find a point $\mathbf{p}_0 \in A$ such that

$$\max_{\mathbf{x} \in A} d(\mathbf{p}_0, \mathbf{x}) = \min_{\mathbf{p} \in A} \max_{\mathbf{x} \in A} d(\mathbf{p}, \mathbf{x}). \tag{16}$$

Several algorithms have been proposed to solve particular cases of this problem. For example, in the case of convex polyhedra, the solution of (16) is the center of the minimum covering sphere for the set of vertices of the polyhedron. The problem of computing the smallest enclosing circle of a set of points in two dimensions was first posed by Sylvester in 1857. At present, we dispose of algorithms for $N = 2$ [12, 27], $N = 3$ [19], and arbitrary N [15, 26].

5.1. AXIS-PARALLEL BOXES OF VARIABLE CENTER AND DIMENSIONS

PROPOSITION 4. *If $A \in \mathcal{H}(R^3)$ is convex, we can find an axis-parallel box C such that*

$$H(C, A) \leqslant \frac{\sqrt{3}}{2}\delta(CA).$$

Proof. Let C be the axis-parallel cube circumscribed to the circumsphere of A, CA.The diameter of this cube is $\sqrt{3}\delta(CA)$.

By Lemma 10, the center of CA, $\mathbf{p}_0$, belongs to A. By Lemma 2, $h(C, A) = d(\mathbf{v}^*, A)$, where $d(\mathbf{v}^*, A) = \max_{i=1,8} d(\mathbf{v}_i, A)(\{\mathbf{v}_i\}$ are the vertices of C), then

$$h(C, A) = d(\mathbf{v}^*, A) \leqslant d(\mathbf{v}^*, \mathbf{p}_0) = \frac{\sqrt{3}}{2}\delta(CA).$$

Since $h(A, C) = 0$, the result follows.

□

PROPOSITION 5. *If $A \in \mathcal{H}(R^3)$ is convex, and C is an axis-parallel box C such that the length L of its largest edge verifies $L > (1+\sqrt{3})\delta(CA)$ then,*

$$H(C, A) > \frac{\sqrt{3}}{2}\delta(CA).$$

Proof. We call F to any of the minimum area faces of C and F' to the opposed face. By PF and PF' we denote the planes containing F and F', respectively. We consider the following cases

- $CA \cap PF \neq \emptyset$ or $CA \cap PF' \neq \emptyset$.
  Assume, for example, $CA \cap PF \neq \emptyset$. The projection of each vertex $\mathbf{v}_{F'}$ of F' on A, verifies

  $$d(\mathbf{v}_{F'}, \pi_A(\mathbf{v}_{F'})) \geqslant L - h,$$

  where h is the distance from PF to a plane T parallel to PF, tangent to CA, and such that CA is included in the half-space delimited by T which does not contain the face F'.
  Since $h \leqslant \delta(CA)$, we have

  $$H(C, A) \geqslant h(C, A) \geqslant d(\mathbf{v}_{F'}, \pi_A(\mathbf{v}_{F'})) \geqslant L - \delta(CA).$$

  Hence

  $$H(C, A) > (1+\sqrt{3})\delta(CA) - \delta(CA) > \frac{\sqrt{3}}{2}\delta(CA).$$

- $CA \cap PF = \emptyset$ and $CA \cap PF' = \emptyset$. CA is placed between the planes PF and PF'.
  Consider the planes T and T' tangent to CA which are parallel to PF and such that there are not points of CA between T and PF and between T' and PF'.
  If we call t to the distance between PF and T and t' to the distance between PF' and T', we have

$$t + t' = L - \delta(CA). \tag{17}$$

  If $\mathbf{v}_F$ and $\mathbf{v}_{F'}$ are vertices of C in F and F', respectively, we have

$$h(C, A) \geqslant d(\mathbf{v}_{F'}, \pi_A(\mathbf{v}_{F'})) \geqslant t',$$
$$h(C, A) \geqslant d(\mathbf{v}_F, \pi_A(\mathbf{v}_F)) \geqslant t,$$

  therefore,

$$h(C, A) \geqslant \max(t, t').$$

  By Lemma 8(b) and (17)

$$\max(t, t') \geqslant \frac{1}{2}(L - \delta(CA)).$$

  Since $L > (1 + \sqrt{3})\delta(CA)$, it follows

$$H(C, A) \geqslant h(C, A) > \frac{\sqrt{3}}{2}\delta(CA).$$

- $CA \cap PF = \emptyset$ and $CA \cap PF' = \emptyset$. CA is not placed between PF and PF'.
  Assume, for example, that PF is between CA and C. Then, for each vertex $\mathbf{v}_{F'}$ of F'

$$d(\mathbf{v}_{F'}, \pi_A(\mathbf{v}_{F'})) \geqslant L.$$

  Then

$$\begin{aligned} H(C, A) &\geqslant h(C, A) \geqslant d(\mathbf{v}_{F'}, \pi_A(\mathbf{v}_{F'})) \geqslant L \\ &> (1 + \sqrt{3})\delta(CA) > \frac{\sqrt{3}}{2}\delta(CA). \end{aligned}$$

$\square$

Now, let the bounding box of A (the minimal axis-parallel box that contains A) be

$$[c_1^1, c_2^1] \times [c_1^2, c_2^2] \times [c_1^3, c_2^3].$$

A reduced search space for the vertex $\mathbf{a}$ of the box $C = \text{conv}(\{\mathbf{a} + \mathbf{b}.\mathbf{d}/\mathbf{b} \in \{0, 1\}^3\})$ is given by the interval

$$I = [c_1^1 - \alpha, c_2^1] \times [c_1^2 - \alpha, c_2^2] \times [c_1^3 - \alpha, c_2^3],$$

where $\alpha \equiv (1 + \sqrt{3})\delta(CK)$.

In effect, Propositions 3 and 4 give necessary conditions for feasible points. Points that do not verify *a priori* one of them can be removed from the search space.

Then, any parallel-axis box C with $\mathbf{a} \notin I$ such that, $C \cap A \neq \emptyset$ (Proposition 3) does not accomplish the condition stated in Proposition 4 because $L > (1 + \sqrt{3})\delta(CA)$ and by Proposition 5

$$H(C, A) > \frac{\sqrt{3}}{2}\delta(CA).$$

By Proposition 5 the search space for the diagonal $\mathbf{d}$ of the box can be choosen as the interval

$$[0, (1 + \sqrt{3})\delta(CA)]^3.$$

5.2. SPHERES OF VARIABLE CENTER AND RADIUS

PROPOSITION 6. *If $A \in \mathcal{H}(R^3)$ is convex, we can find a sphere S, such that*

$$H(S, A) \leqslant \frac{\delta(CA)}{2}.$$

Proof. Let $S = CA$ be the circumsphere of A. By Lemma 10 the center of CA, $\mathbf{p}_0$ belongs to A.

Let $\mathbf{x}$ be an arbitrary point of S not contained in A. Let l be the half-line starting at $\mathbf{p}_0$ and passing through $\mathbf{x}$. Then

$$d(\mathbf{x}, \pi_A(\mathbf{x})) \leqslant d(\mathbf{x}, l \cap \partial A) \leqslant d(l \cap \partial S, \mathbf{p}_0) = \frac{\delta(CA)}{2},$$

hence

$$h(S, A) = \max_{\mathbf{x} \in S} d(\mathbf{x}, A) \leqslant \frac{\delta(CA)}{2}.$$

Since $h(A, S) = 0$, the result follows.

□

PROPOSITION 7. *If* $A \in \mathcal{H}(R^3)$ *is convex, and* S *is a sphere such that* $\delta(S) > \delta(CA) + 2\epsilon$, *then* $H(S, A) > \epsilon$.

Proof. By Lemma 5, the minimum of $h(M(S), CA)$ is attained when $M(S)$ is centered at the circumcenter of A. Call S' to such $M(S)$, then

$$h(S', CA) = \frac{\delta(S) - \delta(CA)}{2} > \epsilon.$$

Since $A \subset CA$, by Lemma 4, it follows that

$$h(S', A) \geqslant h(S', CA),$$

therefore,

$$H(S, A) \geqslant h(S, A) \geqslant h(S', A) \geqslant h(S', CA) > \epsilon.$$

□

Then, if $\delta(S) > 2\delta(CA)$, by Proposition 7, it follows that

$$H(S, A) > \frac{\delta(CA)}{2}$$

and the necessary condition in Proposition 6 is not verified. Therefore, we can vary the radii of the spheres in the interval

$$[0, \delta(CA)].$$

Now, let the bounding box of A be

$$[c_1^1, c_2^1] \times [c_1^2, c_2^2] \times [c_1^3, c_2^3].$$

A reduced search space for the center of the spheres is given by the interval

$$I = [c_1^1 - \beta, c_2^1 + \beta] \times [c_1^2 - \beta, c_2^2 + \beta] \times [c_1^3 - \beta, c_2^3 + \beta],$$

where $\beta = \delta(CA)$.

In effect, if $\mathbf{c} \notin I$ and $S = \bar{B}_r(\mathbf{c})$, the necessary condition $S \cap A \neq \emptyset$ (Proposition 3) is accomplished only if $r > \delta(CA)$ and, then the necessary condition in Proposition 6 is not verified.

5.3. SETS GENERATED BY RIGID MOTIONS OF A COMPACT AND CONVEX SET K_0 IN R^3

PROPOSITION 8. *Let* $A, B \in \mathcal{H}(R^3)$ *and let* $\mathbf{p}$ *be an arbitrary point in* R^3, *then*

$$H(A, B) \leqslant H(A, \{\mathbf{p}\}) + H(\{\mathbf{p}\}, B).$$

Proof. H is a metric in $\mathcal{H}(R^3)$.

□

If $A, K_0 \in \mathcal{H}(R^3)$ are convex and $B = T(K_0)$, we can find a point $\mathbf{p}$ and a translation T such that minimize the upper bound of $H(A, T(K_0))$ in Proposition 8. By Lemma 10, it suffices to consider $T = T_{\mathbf{p}_0 - \mathbf{p}_0'}$ and $\mathbf{p} = \mathbf{p}_0$, where $\mathbf{p}_0$ and $\mathbf{p}_0'$ are the circumcenters of A and K_0, respectively.

In this way, we obtain a reduction of the Hausdorff distance by means of pure translation.

Therefore, a simple procedure for determining a feasible set of displacements is the following:

1. Move K_0 by means of the translation that transforms the circumcenter of K_0 in the circumcenter of A. Call K_0' to $T_{\mathbf{p}_0 - \mathbf{p}_0'}(K_0)$. Consider $\mathbf{p}_0$ as the coordinate origin O.
   Now, by Proposition 3, it suffices to consider only displacements that translate a point in K_0' into a point in A. This set of displacements is difficult to find exactly but it can be approximated in the following way:
2. Compute a point $\mathbf{x}_f$ having maximum norm in K_0'. Consider the sphere

$$B_{\|\mathbf{x}_f\|}(O). \tag{18}$$

   Any rotation which lets O invariant transforms a point of K_0' into an interior point of this sphere.
3. Determine the bounding box I of $B_{\|\mathbf{x}_f\|}(O)$. Let this bounding box be

$$I = I_1 \times I_2 \times I_3 = [a_1^1, a_2^1] \times [a_1^2, a_2^2] \times [a_1^3, a_2^3].$$

4. Compute the bounding box J of A

$$J = J_1 \times J_2 \times J_3 = [b_1^1, b_2^1] \times [b_1^2, b_2^2] \times [b_1^3, b_2^3].$$

5. The approximate feasible set of displacements is given by the interval

$$J - I = (J_1 - I_1) \times (J_2 - I_2) \times (J_3 - I_3) = [b_1^1 - a_2^1, b_2^1 - a_1^1] \\ \times [b_1^2 - a_2^2, b_2^2 - a_1^2] \times [b_1^3 - a_2^3, b_2^3 - a_1^3].$$

(we use the notation given in [28] for the difference of multidimensional intervals).

We remark that

- The choice of the circumcenter, as the reference point for decreasing the Hausdorff distance by translation, is aimed to reduce the size of the bounding box that contains the sphere (18).
  A different problem is the obtainment of a pseudooptimal matching by means of pure translation. This problem has been studied, for example, in [1]. In that paper, the reference point proposed is the Steiner point. Efrat et al. [9] describe algorithms for pseudooptimal matching in the case of sets of points in R^N.
- The above procedure can be applied if we are able to calculate circumcenters, points farthest of the origin, and bounding boxes of the involved figures. In the case of convex polyhedra there exist algorithms for performing all these computations.

6. Conclusion

We prove that some important Hausdorff matching problems can be solved by global Lipschitz optimization techniques. We provide upper bounds of the Lipschitz constant in the cases studied.

Since the results are refered to compact sets, they can be applied to very different matching problems involving:

- Finite sets of points.
- Bounded curves and surfaces.
- Convex and non-convex polyhedra.
- Finite unions of the above type of sets.

We give practical procedures for reducing the search space of the minimization problems when the objective set A and the elements of the approximating family $\mathcal{F}$ are convex.

Although we consider only the case $N = 3$, most part of the results of this paper can be extended, with the corresponding modifications, to any dimension.

Future work should address possible improvements of the Lipschitz constants given here, feasible domains for non-convex sets, obtainment of numerical results, and comparison with other methods.

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